The stability of viscous flow between rotating concentric cylinders with a pressure gradient acting round the cylinders

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(Received 6 February 1959)

The stability of a viscous fluid between concentric cylinders is analysed, for the case in which the basic velocity distribution is the sum of a velocity distribution due to the rotation of the cylinders (Taylor 1923), and a 'pumping' velocity distribution due to a pressure gradient acting round the cylinders (Dean 1928). The critical Taylor number is computed for a wide range of values of the ratio of average velocity of pumping to average velocity of rotation for the case in which the outer cylinder is stationary. It is assumed that the spacing between the cylinders is small.

1. Introduction

The stability of a viscous flow between two concentric rotating cylinders was first considered experimentally and theoretically by Taylor (1923). Using the assumption that the spacing between the cylinders is small compared to the mean radius, he obtained a criterion for the onset of a secondary motion with a cellular form which was verified by his experiments. The eigenvalue problem which arises in this analysis is of considerable interest, and has been discussed by several authors, including Pellew & Southwell (1940), Chandrasekhar (1954*a*) and DiPrima (1955).

A similar type of instability occurs when a viscous fluid flows in a curved channel under a pressure gradient acting round the channel. This problem was first considered by Dean (1928), for the case in which the annulus is thin compared to the mean radius. The resulting eigenvalue problem has also been studied by Reid (1958) and Hämmerlin (1958). Their results verify those of Dean (1928).

In a number of engineering applications, stability problems of the type just described are of considerable interest. One such problem, which has been considered recently by Brewster & Nissan (1958) and Brewster, Grosberg & Nissan (1959), is that of the stability of a viscous fluid between concentric cylinders when the inner cylinder is rotating (the Taylor problem) and at the same time the fluid is being pumped round the annulus (the Dean problem). The pumping may be in the direction of the rotation or opposed to it. The theoretical conclusions of Brewster *et al.* (1959) are based on the use of the necessary condition for instability that the square of the circulation should decrease outwards from the inner cylinder. The eigenvalue problem is explicitly solved only in the case when the average velocity of rotation is equal to, but opposite in direction to, the average velocity of pumping. In general their conclusions are supported by their experimental results.

In this paper the stability problem for the above type of velocity distribution is formulated. In the case when the spacing between the cylinders is small compared to the mean radius, the eigenvalue problem is solved by the methods first suggested by Chandrasekhar (1954*a*). Numerical results for the critical value of the parameter governing stability are given for a wide range of ratios of average velocity of pumping to average velocity of rotation. The results are in good agreement with the experimental data of Brewster *et al.* (1959).

2. The disturbance equations

Let (r, θ, z) be cylindrical co-ordinates, with the z axis coinciding with the axis of the cylinders, and let R_1, R_2, Ω_1 and Ω_2 denote the radii and angular velocities of the inner and outer cylinders, respectively. If u_r, u_θ, and u_z denote the components of velocity in the increasing r, θ and z direction and p denotes the pressure, the Navier–Stokes equations admit a steady solution of the form

$$u_r = u_z = 0, \quad u_\theta = V(r), \quad \frac{\partial p}{\partial r} = f(r). \tag{1}$$

Now superimpose on this steady motion a small disturbance of a form such that the θ-component of velocity is

$$u_\theta(r,\theta,z,t) = V(r) + v(r)\cos\lambda z\, e^{\sigma t}. \tag{2}$$

The motion will be stable if the real part of σ is less than zero, and unstable if it is greater than zero. It can be anticipated, from similar problems and the experimental results, that for the problem studied here the secondary motion will be of a stationary cellular nature. Hence, we shall only consider the marginal state $\sigma = 0$. In this case the linearized equations for the disturbance velocities are

$$\left.\begin{aligned}(L-\lambda^2)^2 u &= \frac{2\lambda^2}{\nu}\frac{1}{r}V(r)\,v,\\ (L-\lambda^2)\,v &= \frac{1}{\nu}\left(\frac{dV}{dr}+\frac{V}{r}\right)u,\end{aligned}\right\} \tag{3}$$

where u is the disturbance velocity in the r direction, ν is the kinematic viscosity, and

$$L = \frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}.$$

In the case that $d = R_2 - R_1$ is small compared with $\frac{1}{2}(R_1+R_2)$, the operator L can be replaced by d^2/dr^2, and V/r and $dV/dr + V/r$ can be approximated by

$$V(r)/r = \Omega_1\{1-(1-k)\,x\} + \frac{d^2}{2\mu R_1^2}\frac{\partial p}{\partial \theta}(x^2-x), \tag{4}$$

$$\frac{dV}{dr}+\frac{V}{r} = 2A + \frac{d}{2\mu R_1}\frac{\partial p}{\partial \theta}(2x-1), \tag{5}$$

where $k = \Omega_2/\Omega_1$, $r = R_1 + dx$, $A = \Omega_1\{1-(kR_2^2/R_1^2)\}/\{1-(R_2^2/R_1^2)\}$, and μ is the viscosity. These approximations are correct up to terms $O(d/R_1)$. The first term on the right of equation (4) represents the velocity profile due to the rotation of the cylinders, and the second term represents the velocity profile due to pumping. Using equations (4) and (5), the system of differential equations (3) becomes

$$(D^2-a^2)^2 u = \{1-(1-k)x-Q(x^2-x)\}v, \tag{6}$$

$$(D^2-a^2)v = -a^2T\left\{1+\frac{Q}{1-(kR_2^2/R_1^2)}(2x-1)\right\}u, \tag{7}$$

where u has been redefined as $(\nu/2a^2\Omega_1 d^2)u$ and

$$\left.\begin{aligned} D = \frac{d}{dx}, \quad a = \lambda d, \quad T = -\frac{4A\Omega_1 d^4}{\nu^2}, \quad Q = 3(1+k)\frac{V_P}{V_R}, \\ V_R = R_1\Omega_1\frac{1+k}{2}, \quad V_P = -\frac{d^2}{12\mu R_1}\frac{\partial p}{\partial \theta}. \end{aligned}\right\} \tag{8}$$

The quantities V_R and V_P are the average velocities due to rotation and pumping, respectively. The parameter, T, is commonly referred to as the Taylor number.

The requirement of no slip at the boundaries gives the boundary conditions

$$u = v = Du = 0 \tag{9}$$

at $x = 0$ and $x = 1$.

The system of equations (6) and (7), together with the boundary conditions, determine an eigenvalue problem for T as a function of k, R_2/R_1, Q and a. For fixed values of k, R_2/R_1, and Q, the minimum value of T with respect to a determines the critical value, T_c, at which instability will first set in. Notice that when the pumping is zero, i.e. $Q = 0$, equations (6) and (7) reduce to the Taylor problem as treated by Chandrasekhar (1954*b*). In order to obtain the problem of a viscous flow in a curved channel under a pressure gradient acting round the channel, i.e. $Q \to \infty$, it is necessary to redefine u as u/Q, and then let $Q \to \infty$, noting that the combination TQ^2 is $72(V_P d/\nu)^2/(d/R_1)$.

The determination of T_c for the general case is a four parameter problem, and would require a considerable number of computations. We shall now restrict ourselves to the physically interesting case (see Brewster & Nissan (1958) or Brewster *et al.* (1959)) when the outer cylinder is at rest.

3. The outer cylinder at rest

In this case $k = 0$ and equations (6) and (7) reduce to

$$(D^2-a^2)^2 u = \{1-x-Q(x^2-x)\}v, \tag{10}$$

$$(D^2-a^2)v = -Ta^2\{1+Q(2x-1)\}u. \tag{11}$$

Thus T is now defined as a function of a and $Q = 3V_P/V_R$; and for fixed values of Q the minimum value of T with respect to a gives the critical Taylor number, T_c, at which instability will occur.

The method that was used in obtaining approximate values of T_c and a_c is completely analogous to that used by Chandrasekhar (1954*a*) in treating similar problems. The function $v(x)$ is expanded in a set of complete functions satisfying $v(0) = v(1) = 0$. In this case

$$v(x) = \sum_{m=1}^{\infty} A_m \sin m\pi x, \tag{12}$$

and with this expansion for $v(x)$, equation (10) is solved for $u(x)$.

The four constants of integration that appear in the solution are determined by the conditions $u = Du = 0$ at $x = 0$ and 1. The functions u and v are then substituted in equation (11), the result is multiplied by $\sin n\pi x$, $n = 1, 2, \ldots$ and integrated from 0 to 1. This leads to infinitely many equations for the A_m. If all of the A_m are not to vanish identically the determinant of the coefficients must be zero, and this gives a relation between T, a and Q. The details of the analysis, while straightforward, are quite lengthy and are given in the Appendix.

Q	$a = \lambda d$	T/π^6	N	T_{N-1}/T_N
30	3·76	0·085	3	0·982
21	3·70	0·155	—	0·984
15	3·60	0·266	—	0·987
10	3·45	0·474	—	0·992
6	3·30	0·866	—	0·996
3	3·14	1·54	—	1·006
1·0	3·13	2·53	—	1·006
0·50	3·13	2·96	—	1·005
0	3·12	3·53	—	1·004
− 0·50	3·17	4·35	—	1·002
− 1·0	3·24	5·64	—	0·9997
− 1·5	3·40	7·97	—	0·9992
− 2·0	3·80	13·1	—	1·02
− 2·5	5·0	24·7	4	1·002
− 2·75	5·73	33·1	—	1·016
− 3·00	6·35	42·6	—	1·045
− 3·25	7·05	53·7	—	1·094
− 3·50	7·4	66·4	—	1·109
− 3·75	5·8	64·5	—	0·949
− 4·00	5·5	47·8	—	0·969
− 4·50	5·37	29·1	—	0·981
− 5·00	5·20	19·0	—	0·987
− 6·00	5·00	10·0	—	0·995
− 8·00	4·70	4·00	3	1·02
− 10·00	4·55	2·10	—	1·01
− 15·00	4·35	0·720	—	0·984

TABLE 1. Critical Taylor Numbers and corresponding values of a for assigned values of Q. (N = number of terms used in approximation)

The numerical results for T_c and a_c for a range of values of Q from -15 to 30, i.e. V_P/V_R ranges from -5 to 10, are tabulated in table 1, and shown graphically in figures 1 and 2. It is clear from table 1 that for most values of Q a three-term approximation gives satisfactory results; however, for the range $-8 < Q < -2$ it was necessary to use another term in the approximation.

For $Q = 0$ the values of T_c and a_c are in agreement with those of Chandrasekhar (1954*b*). When the average velocities of pumping and rotation are equal but in opposite directions, i.e. $Q = -3$, the values of $T_c/\pi^6 = 42·6$, and $a_c = 6·35$ may be compared with the values 40·7 and 5·7* found by Brewster *et al.* (1959) using a six-term approximation and following the methods of Dean (1928). It is impossible to compare the present results with the results for zero rotation;

* In part, the poor agreement for a_c probably follows from the fact that, in the neighbourhood of the minimum value, T is very insensitive to changes in a and hence an accurate determination of a_c is quite difficult.

however, as $Q \to \pm\infty$ it appears from figure 2 that a_c is approaching a value of about 4, and Reid (1958) gives $a_c = 3{\cdot}96$ for this case.

The experimental results* given by Brewster *et al.* (1959) have been used to compute values of T_c/π^6 which are shown in figure 1. As can be seen the experimental points are in good agreement with the theoretical curve obtained here.

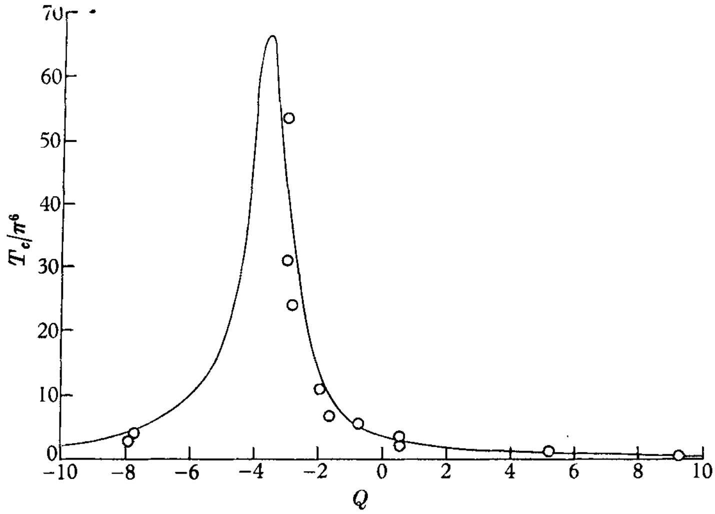

FIGURE 1. The variation of the critical Taylor number, T_c, as a function of $Q = 3V_P/V_R$. Values of T_c computed from the experimental data of Brewster *et al.* (1959) are also shown. ○, experimental points, Brewster *et al.* (1959).

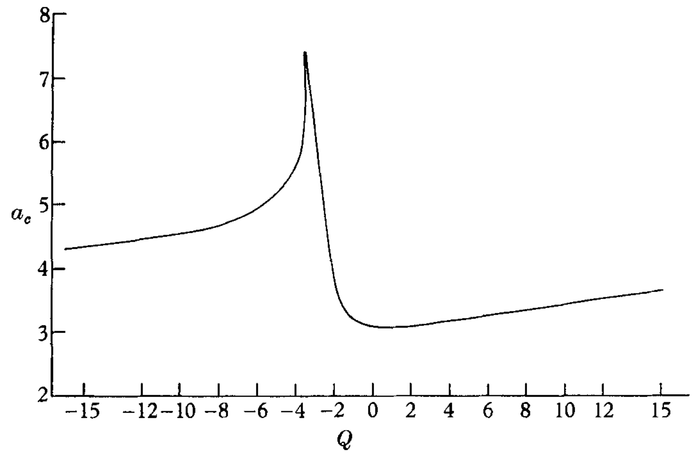

FIGURE 2. The variation of the critical wave-number a_c as a function of $Q = 3V_P/V_R$.

* The experiments were run with a glycerine solution as the fluid. The value of R_1 was 15·6 cm and the value of d was 0·858 cm, which was determined by making the experimental results agree with the theoretical results of Taylor for $Q = 0$. Some experiments were also run with a water-dye solution for the range of value of $-5 < Q < -3$, but there was difficulty in making accurate observations in this case and the scatter of the results was quite large. However, in a private communication, Dr Grosberg of the University of Leeds has informed the author that the average values of the latter results do show good agreement with the theoretical curve given in figure 1.

Finally, for an assigned ratio of V_P/V_R, the curve shown in figure 1 may be used to compute the critical value of V_R at which instability will occur. Also, for given values of V_P or V_R the corresponding ranges of V_R and V_P for which the flow is stable may be computed. It should be noted that in the neighbourhood of $V_P/V_R = -1$ the value of T_c is quite large. This means that as long as the average velocities of pumping and rotation are nearly equal and opposite in sign, and hence tend to annul each other on the average, they may be quite large in magnitude before instability will occur.

The results presented in this paper were obtained in the course of research sponsored by the Mechanics Branch of the Office of Naval Research, Washington. The author would like to express his appreciation to Prof. A. H. Nissan for suggesting this problem and for his continued interest throughout the course of the analysis. Thanks are also due to Mr Burt Liebowitz who carried out the rather lengthy numerical computations.

Appendix

The solution of equation (10), corresponding to the expansion of $v(x)$ given in equation (12), that satisfies the boundary conditions $u = Du = 0$ at $x = 0$ and 1 is

$$u(x) = \sum_{m=1}^{\infty} \frac{A_m}{(m^2\pi^2+a^2)^2}\{A_1^{(m)}\cosh ax + B_1^{(m)}\sinh ax + A_2^{(m)}x\cosh ax + B_2^{(m)}x\sinh ax$$
$$+[1+f_m-(1-Q)x-Qx^2]\sin m\pi x - e_m[(1-Q)+2Qx]\cos m\pi x\},$$

where
$$f_m = \frac{4(a^2-5m^2\pi^2)}{(m^2\pi^2+a^2)^2}, \quad e_m = \frac{4m\pi}{m^2\pi^2+a^2},$$

$$A_1^{(m)} = (1-Q)\,e_m,$$

$$B_1^{(m)} = \frac{m\pi}{\Delta}\{a\alpha_m + (\sinh a + a\cosh a)\beta_m - (\sinh a)\gamma_m\},$$

$$A_2^{(m)} = -\frac{m\pi}{\Delta}\{(\sinh a)^2\alpha_m + (a\sinh a + a\cosh a)\beta_m - (a\sinh a)\gamma_m\},$$

$$B_2^{(m)} = \frac{m\pi}{\Delta}\{(\sinh a\cosh a - a)\alpha_m + (a^2\sinh a)\beta_m - (a\cosh a - \sinh a)\gamma_m\};$$

and
$$\alpha_m = 1+QF_m, \quad \beta_m = -G_m(-1)^{m+1}\{(1+Q)+(1-Q)\cosh a\},$$
$$\gamma_m = (-1)^{m+1}QF_m - (1-Q)(a\sinh a)\,G_m, \quad \Delta = \sinh^2 a - a^2,$$

where
$$F_m = \frac{12(m^2\pi^2-a^2)}{(m^2\pi^2+a^2)^2}, \quad G_m = \frac{4}{m^2\pi^2+a^2}.$$

Substituting for u and v in equation (11) multiplying the result by $\sin n\pi x$, $n = 1, 2, \ldots$, and integrating from 0 to 1 gives the following system of equations:

$$\sum_{m=1}^{\infty}\left\{K_{nm}(a,Q) - \delta_{nm}\frac{(n^2\pi^2+a^2)^3}{2a^2T}\right\}\bar{A}_m = 0 \quad (n = 1, 2, \ldots),$$

where $\bar{A}_m = A_m/(m^2\pi^2+a^2)^2$, $\delta_{nm} = 0$ if $n \neq m$ and $= 1$ if $n = m$, and the lengthy expression $K_{nm}(a,Q)$ is given by

$$K_{nm} = \frac{n\pi}{n^2\pi^2+a^2}\{g_n A_1^{(m)} + h_n B_1^{(m)} + k_n A_2^{(m)} + l_n B_2^{(m)}\} + \frac{\delta_{nm}}{2} - X_{nm} + QY_{nm} + Q^2 Z_{nm}.$$

In this expression

$$g_n = (1-Q)\epsilon_n + 2Q\zeta_n, \qquad k_n = (1-Q)\zeta_n + 2Q\theta_n,$$
$$h_n = (1-Q)\lambda_n + 2Q\mu_n, \qquad l_n = (1-Q)\mu_n + 2Q\nu_n,$$

with

$$\epsilon_n = 1 + (-1)^{n+1}\cosh a, \qquad \theta_n = \epsilon_n H_n - 2E_n\lambda_n - 1,$$
$$\lambda_n = (-1)^{n+1}\sinh a, \qquad \nu_n = \lambda_n H_n - 2E_n(\epsilon_n - 1),$$
$$\zeta_n = \epsilon_n - 1 - E_n\lambda_n, \qquad E_n = \frac{2a}{n^2\pi^2+a^2},$$
$$\mu_n = \lambda_n - E_n\epsilon_n, \qquad H_n = 1 - \frac{1}{a}E_n + 2E_n^2,$$

and

$$X_{nm} = \begin{cases} 0, & n+m \text{ even}, \ n \neq m, \\ \frac{1}{4}, & n = m, \\ -4A_{nm} + 2B_{nm}e_m, & n+m \text{ odd}; \end{cases}$$

$$Y_{nm} = \begin{cases} -12A_{nm} + 4B_{nm}e_m, & n+m \text{ even}, \ n \neq m, \\ \dfrac{3}{4n^2\pi^2} + \dfrac{1}{\pi n}e_n - \frac{1}{2}f_n, & n = m, \\ -4A_{nm}, & n+m \text{ odd}; \end{cases}$$

$$Z_{nm} = \begin{cases} 0, & n+m \text{ even}, \ n \pm m, \\ 0, & n = m, \\ 4A_{nm} - \dfrac{12}{\pi^4}\left[\dfrac{1}{(n-m)^4} - \dfrac{1}{(n+m)^4}\right] + e_m\left\{-2B_{nm} + \dfrac{8}{\pi^3}\left[\dfrac{1}{(n-m)^3} + \dfrac{1}{(n-m)^3}\right]\right\} & \\ \quad + 8A_{nm}f_m, & n+m \text{ odd}; \end{cases}$$

where

$$A_{nm} = \frac{nm}{\pi^2(n^2-m^2)^2}, \qquad B_{nm} = \frac{n}{\pi(n^2-m^2)}.$$

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